Journal of Geometry and Physics 8 (1992) 327–334
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Searching for integrability

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Received 26 August 1991

The self-dual Yang–Mills equations in four dimensions are integrable, for any gauge group, via the twistor transform. Integrability of the Yang–Mills equations proper in four dimensions with a finite dimensional gauge group cannot reasonably be hoped for. However, longstanding questions about the large N limit of QCD suggest that a new form of integrability might conceivably emerge in the limit of an infinite dimensional gauge group.

Keywords: *Yang–Mills equations, integrability*
1991 MSC: 81T13

The twistor transform, which Roger Penrose introduced and whose whole development he has so influenced, lies at the heart of many mysteries. It exhibits the self-dual Einstein [1] and Yang–Mills [2] equations as what one might roughly call integrable equations. These (and some analogs discovered by twistor methods) are the only known nonlinear equations in four dimensions that might reasonably deserve that name. Twistorial investigations of these equations, and their various offspring and dimensional reductions, have paid recurrent dividends that show no sign of diminishing.

The twistor explanation of integrability of the self-dual equations is so incisive that it might well be regarded as the model of what an explanation of integrability should be. To appreciate the significance of this, note that various kinds of integrable models, which at first sight might look like exotica, play a key role in a startling range of problems. In recent years, integrable systems have been used, for instance, to construct invariants of knots and three manifolds, to describe the geometry of the moduli space of Riemann surfaces, and to construct classical solutions of string theory – just to mention a few problems that are more or less related. Deeper study of some of these problems may well lead back to twistor theory and the self-dual equations in four dimensions.

* Research supported in part by NSF Grant PHY86-20266.

Further progress in twistor theory is bound to involve making contact with developments in more conventional quantum field theory. I would like to offer a few reflections about a possible area of future contact between twistor theory and conventional quantum field theory. In doing so, I decided to focus on an important problem that is not already known to be connected with integrability but where there is reason to suspect that such a connection might be possible. The problem in question has not been much considered in the light of twistor theory, though I myself have tried to do so to some extent. I do not have much to say that is really new, but I hope that these reflections will be of some interest.

Integrability in Yang–Mills theory

Much interest in twistor theory has focussed on trying to find an analog of the twistor construction for the full-fledged Einstein and Yang–Mills equations (not their self-dual versions). At least in the Yang–Mills case, there is indeed an interpretation of the equations as integrability equations, roughly, on slightly thickened light-like lines [3, 4]. This then leads to a correspondence between solutions of the Yang–Mills equations and holomorphic vector bundles on a suitable space of thickened null lines.

This non-self-dual construction was found by imitating the twistor transform of the self-dual Yang–Mills equations, and incorporates some of the geometrical ideas of the latter. In particular, it incorporates Penrose's lovely idea of transforming local differential geometric data into global complex analytic data. On the other hand, the non-self-dual construction certainly lacks much of the beauty and power of the original twistor transform.

This in itself should come as no surprise. The twistor transform of the self-dual Yang–Mills equations exhibits their integrability, but there is every reason to expect, even without any calculation, that the Yang–Mills equations proper will not be integrable in four dimensions. Integrable models in four dimensions are highly constrained, for instance to have trivial scattering data [5]. (This is usually formulated as a quantum statement, but the arguments, modulo technicalities that I think are inessential, also apply classically.) The phenomena described by the four dimensional Yang–Mills equations are far too complex to be described by an integrable system.

What should we aim for instead? This question almost answers itself if one asks what integrability of the Yang–Mills equations would be good for if – contrary to expectations – it were discovered. Such a discovery undoubtedly would be brought to bear on the longstanding problem of understanding QCD, the presumptive theory of strong interactions.

The strong interactions are believed to be described by an SU(3) quantum gauge theory in four dimensions. This theory has been extensively tested by numerous experiments that verify those of its predictions that follow either from global symmetries (spontaneously broken or not) or from asymptotic freedom. On the other hand, because of the strong coupling of QCD at large distances, much remains obscure; in particular, the predictions of the theory concerning particle masses, magnetic moments, etc., are out of reach except in enormous Monte Carlo computer calculations. Such calculations are unsatisfying as a basis for understanding, but have given important further evidence that QCD is correct. At an even more fundamental level, the key theoretical mysteries of color confinement, dynamical mass generation, and spontaneous chiral symmetry breaking have been far out of reach of analytic understanding.

If we would learn that the Yang–Mills equations were integrable classically, we would try (in the spirit of the quantum inverse scattering method in two dimensions [6]) to find a corresponding integrability of QCD at the quantum level.

This is impossible; the phenomena described by QCD are far too complex for QCD to be an integrable system. In particular, the S matrix of QCD is not 1. What can we hope for instead?

Although QCD is not an integrable system with the usual $SU(3)$ gauge group, 't Hooft showed long ago (see ref. [7]; I reviewed the arguments some years later [8]) that, if one replaces the SU(3) gauge group by SU(N) (and makes a few qualitative assumptions, like the assumption that color confinement persists), then in the limit of $N \to \infty$ QCD simplifies radically, becoming a weakly coupled theory in which the basic glueball coupling constant is of order $1/N$. If there is any way to understand QCD analytically, this must be it, since the key theoretical mysteries (color confinement, mass generation, chiral symmetry breaking, infinite hierarchy of hadron masses) that one wants to understand apparently all persist in this limit while the residual hadron interactions that otherwise ensure that the problem is intractable (unintegrable) disappear. Nature has a way of treating kindly physicists who find approximation schemes that are qualitatively correct, and so there is a good chance that a solution of QCD for $N \to \infty$ would not only solve the theoretical mysteries but also give for added measure a good quantitative account of the hadron world.

Unfortunately, the arguments that show that QCD becomes free for $N \to \infty$ are far too crude to say anything else (beyond a few general relations that are pretty well obeyed in nature) about the limiting large N theory. In particular, one does not know anything about the masses, spins, and quantum numbers of the hadrons of the large N limit, much less how to compute the contributions of order $1/N$ to their interactions. We have no idea whether the limited simplification found by 't Hooft is all of the simplification there is, or whether it is just the tip of the iceberg of an "integrability" that would really make the large N limit calculable. I put quotes around the word "integrability" because a precise and suitable definition of this term is hard to give, and some of the usual definitions might not apply here. What we are hoping is that there is a major, not obvious simplification of the large N limit of QCD, which will make it tractable, and which one might loosely describe as "integrability".

I suppose that the main hint that this might be the case is that the $1/N$ expansion of QCD is organized by an expansion in Riemann surfaces, on which Feynman diagrams are drawn. This gives a striking analogy (already noted by 't Hooft) of the $1/N$ expansion with string theory. $1/N$ plays the role of the closed string coupling constant. Many physicists have suspected that the large N limit of QCD is integrable because it can be understood as some presently unknown kind of string theory. Presently there is no way to know whether the analogy between string theory and the $1/N$ expansion of QCD is a frustrating accident or is a trace of something of this kind. I personally doubt that the methods that have been brought to bear on the problem up to now are close to adequate, so we will probably be in the dark for some time.

Thus, although classical Yang–Mills field theory cannot reasonably be an integrable system in four dimensions, nor can the quantum theory be integrable for a generic gauge group, it is entirely possible that in a suitable sense four dimensional Yang–Mills theory becomes integrable for $N \to \infty$. This is usually stated (along the lines of the last paragraph) as something that might be true in the quantum theory, but I personally hope that if true at all it is true in a suitable sense also in the classical theory. If so, ideas of twistor theory, perhaps even the known twistor-like transform based on integrability on light-like lines, may well be relevant.

If this optimistic scenario is valid, then to really achieve integrability, one presumably should set $N = \infty$ in the classical theory. I do not know which version of $SU(\infty)$ one should consider. However, for all of the reasons that I have tried to sketch, I think that if one wants to find more integrability in four dimensional gauge theories than is already known, the large N limit is the right place to look. It is encouraging that Ward has shown [9] that the Nahm equations (one of the numerous integrable reductions of the self-dual Yang–Mills equations!) can be solved much more explicitly for certain infinite dimensional gauge groups than for finite dimensional ones. (Ward, following ref. [10], takes the gauge group to be the group of area preserving diffeomorphisms of a two dimensional surface. It is not at all clear whether this is the right version of $SU(\infty)$ for understanding the large N limit of QCD.) Also, the ADHM construction of instantons [11] seems to be much more tractable for $N \to \infty$ than for any finite N. (For any given N, and sufficiently large instanton number k, one runs into intractable quadratic equations, but in the large N limit, for any given k, there is no such problem.)

The only work I know using the interpretation of the Yang–Mills equations via light-like integrability to actually describe some solutions of the equations was by Manin [12, pp. 253–267]. His construction (involving a generalization of the ADHM construction) gives non-vacuous results for SU(N) gauge groups with $N \geq 66$. (See p. 266 for the statement of this restriction.) The restriction to $N \geq 66$ may make the construction look exotic, but that is the wrong attitude. The question that we should be asking is precisely whether some construction, which might well turn on only for sufficiently large N, could give a universal description of the Yang–Mills solutions for $N \to \infty$.

One way to understand how the twistor construction of the self-dual Yang–Mills equations exhibits their integrability is the following. Consider a line $X \cong \mathbb{CP}^1$ in twistor space (corresponding to a point in Minkowski space). X can be covered with two very simple open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ (for instance, two copies of a disc or of $\mathbb{C}$), and so a holomorphic vector bundle on X can be described by giving a single transition function f on $\mathcal{O}_1 \cap \mathcal{O}_2$, which is required only to be invertible. Extending this a bit, a holomorphic vector bundle on a small neighborhood of X in twistor space can likewise be described by a single transition function f. This exhibits the integrability of the self-dual Yang–Mills equations: vector bundles on twistor space and therefore the solutions of the self-dual equations are constructed from the freely specifiable function f. Technically, the reason that the relation of the full-fledged Yang–Mills equations to vector bundles on the space of thickened null lines does not lead to integrability is that in this case $\mathbb{CP}^1$ is replaced by $\mathbb{CP}^1 \times \mathbb{CP}^1$. $\mathbb{CP}^1 \times \mathbb{CP}^1$ cannot be covered with two elementary open sets, so a holomorphic vector bundle on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is not naturally described in terms of a single transition function, but requires a collection of transition functions f_{ij}, which must obey an awkward cocycle relation. To exhibit integrability of the $N = \infty$ Yang–Mills equations, one might try to imitate Ward's work on the Nahm equation and find some simplification in the description of holomorphic vector bundles on (neighborhoods of) $\mathbb{CP}^1 \times \mathbb{CP}^1$, for appropriate infinite dimensional gauge groups. In trying to do this, one does not necessarily want to naively describe the bundle by giving a collection of transition functions. Indeed, much of the power of the twistor transform comes from the fact that there are many possible ways of describing holomorphic bundles.

Some simpler analogs

It could be that at present four dimensional Yang–Mills theory is too difficult and that we should concentrate on simpler cases. The most obvious simplifications are achieved by reducing the number of dimensions. In two dimensions, non-abelian gauge theories are "trivial" in the absence of matter, in the sense that the classical field equations reduce to the statement that $D_\alpha F = 0$ (the curvature is covariantly constant). The solutions of this equation are trivially classified locally and are really interesting only globally. In fact, globally, the classical Yang–Mills problem in two dimensions has been used to understand the topology of the moduli space of flat connections (or holomorphic vector bundles) on a surface [13]. The corresponding quantum field theory, which is similar in flavor to topological field theory, is again mainly of global interest. It is "trivial" enough that it can be understood pretty well for any gauge group. It leads to formulas for the volumes of moduli space of flat connections [14].

A more "physical" problem, much more similar to four dimensional QCD, arises if we couple matter fields, say fermions in the N dimensional representation of SU(N), to two dimensional quantum gauge fields. In this case, we get a theory which is quite intractable for a finite dimensional gauge group, but beautifully soluble in the limit of $N \to \infty$ (as 't Hooft showed in the original work). In fact, after so many years this is still the nicest example in which the large N limit has been used to shed light on dynamics of quantum gauge theories.

A big jump in difficulty comes if we go to three spacetime dimensions. Our question about whether integrability arises for $N \to \infty$ could well be raised here. No more is known about integrability of the three dimensional classical Yang–Mills equations than about the four dimensional case. At the quantum level, the fact that three dimensional Yang–Mills theory is superrenormalizable (rather than asymptotically free, as in four dimensions) appears to be a big potential simplification, but in practice it has not been exploited very successfully. Even in the large N limit, three dimensional quantum Yang–Mills theory is as little understood as the four dimensional case. Twistorians might want to begin there.

Hamiltonians and Lagrangians

In broad terms, there are two approaches to understanding a quantum field theory – the Hamiltonian approach and the Lagrangian approach.

In the Hamiltonian approach, one considers a classical phase space $\mathcal{P}$, which can be described covariantly as the space of all classical solutions of the theory under study, up to gauge transformation. $\mathcal{P}$ carries a natural symplectic structure. One must quantize it, construct the Hamiltonian operator, and attempt to diagonalize it approximately. A twistor construction exhibiting integrability of the $N \to \infty$ Yang–Mills equations in three or four dimensions would presumably give a new description of $\mathcal{P}$; if the symplectic structure and the classical Hamiltonian were visible in this language, one could aim to use this as the starting point for quantization.

In the Lagrangian approach, one considers the space $\mathcal{W}$ of all classical fields in spacetime (not necessarily classical solutions), up to gauge transformation. In Yang–Mills theory, $\mathcal{W}$ has a natural Riemannian metric, and formally this gives a Riemannian measure $\mathrm{d}\mu$. One introduces a natural function on this space, the Lagrangian L, and one aims to carry out the Feynman path integral over $\mathcal{W}$:

$$Z = \int_{\mathcal{W}} \mathrm{d}\mu \ \mathrm{e}^{-L} . \tag{1}$$

To carry out such a program in twistor theory, one would need a twistorial description of $\mathcal{W}$ in which the function L would be visible. For instance, let M be a (complexified) spacetime of any dimension. Let $\mathcal{Q}$ be the space of complex null lines (ordinary ones, not thickened ones; we could also consider all lines rather than null ones). Then by imitating the original twistor transform, gauge fields on M are in one to one correspondence with holomorphic vector bundles on $\mathcal{Q}$ obeying certain conditions. If one could describe the standard Yang–Mills Lagrangian

$$L = \int_M \mathrm{d}^n x \ \mathrm{Tr} F \wedge *F \tag{2}$$

in that language – and I do not know any reasonably attractive way to do so – then one could try to give a twistorial formulation of the Feynman path integral (1). Of course, one would expect it to be intractable except possible for $N \to \infty$.

It is interesting to note that if M is four dimensional and N is a three dimensional Cauchy hypersurface, then morally speaking instantons on M are the same as gauge fields on N. (A gauge field on N provides initial data for an instanton on M.) Instantons on M can be described by the ADHM construction, which was originally formulated globally and has its greatest power in that context. However, the ADHM construction can also be described locally [15]; this is relevant to the Nahm construction of monopoles [16]. The ADHM construction of instantons on M could possibly be interpreted as a description of the space $\mathcal{W}$ of gauge fields on N and so used as a starting point for discussing the Feynman path integral of three dimensional Yang–Mills theory. It is important here to use a local version of the ADHM construction so that one is not limited to particular initial data on N that lead to global instantons on M. To proceed down this road, one would again need to discover a reasonable formula in the ADHM variables for the Lagrangian functional on $\mathcal{W}$.

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